A Relational Abstraction for Functions

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Overview. This paper concerns the abstraction of sets of functions for use in abstract interpretation. The paper gives an overview of abstract domains for functions and formalizes a new family of relational abstract domains that allows sets of functions to be abstracted more precisely than with known approaches, while being still machine-representable.

A major strength of Abstract Interpretation is the ability create complex abstract domains from simpler ones [1]. In particular, (i) Galois connections can be composed, which allows a complex abstraction to be described as the composition of simpler ones, offering the ability to identify clearly the different kind of approximations that take place; (ii) given two abstractions for sets of elements, $\wp(D_i)$, $i = 1, 2$, there exist techniques for abstracting functions of signature $D_1 \rightarrow D_2$ [2].

The starting point for the paper is the abstraction method defined in [5], which presents a family of abstract domains that are useful when it is desired to connect storage elements (e.g., elements of arrays and lists) with numeric quantities. This paper reformulates that abstraction in a more general way—as a general method of abstracting a set of functions—which allows the basic idea from [5] to be applied more widely. Moreover, when the new formulation is compared with previously known ways of abstracting a set of functions, it yields more precise abstractions. We are just beginning to explore instantiations of the method that go beyond the ones used in [5].

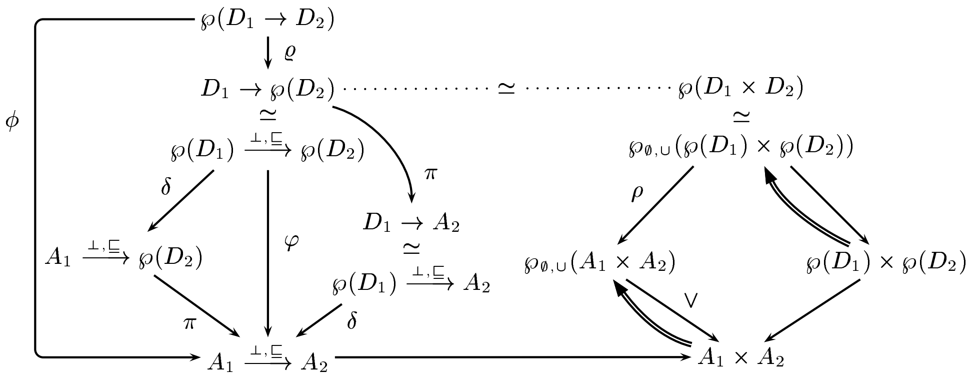
We formalize a generic abstract-interpretation combinator, which abstract sets of functions of signature $D_1 \rightarrow D_2$ in a relational way, assuming the existence of abstractions A_1 and $A_2[n]$ for $\wp(D_1)$ and $\wp((D_2)^n)$, respectively (where A_1 is of finite cardinality n). The obtained abstract domain is precisely $A_2[n]$. In contrast to A_1, $A_2[n]$ may be a complex lattice (relational, infinite, and of infinite height), like the lattice of octagons [4] or convex polyhedra [3]. This relational function-abstraction is more precise than the classical approach described in the literature [2], because of its ability to represent relationships between the images of different elements mapped by a set of functions.

In terms of precision, the relational function-abstraction $A_2[n]$ lies in-between the classical function-abstraction $A_1 \rightarrow A_2$ ($\simeq (A_2)^n$) and its disjunctive completion. The important point is that $A_2[n]$ is still finitely representable in the same circumstances as $A_1 \rightarrow A_2$ (i.e., when $A_1 \rightarrow A_2$ is finitely representable).

Classical abstractions of functions and relations. Fig. 1 depicts the classical methods for abstracting functions of signature $D_1 \rightarrow D_2$ and relations $R \subseteq D_1 \times D_2$, that can be built from two Galois connections $\wp(D_1) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}}$ and $\wp(D_2) \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} A_2$. These abstractions are described in [2] and we use the same notation. Some of the abstractions are illustrated on Fig. 3, where $D_1 = U$,

$A_1 = (U^\sharp)^\top_\bot$ (a flat lattice completed by $\top$ and $\bot$), $D_2 = \mathbb{R}$, and $A_2 = \wp(D_2)$ (no abstraction of the codomain in the example).

In summary, ϱ abstracts a set of functions $F \subseteq D_1 \to D_2$ by a single function $F^\sharp : D_1 \to \wp(D_2)$. One can then abstract the equivalent transformer $F^\sharp : \wp(D_1) \xrightarrow{\bot,\sqsubseteq} \wp(D_2)$ with a function $F^{\sharp\sharp} : A_1 \xrightarrow{\bot,\sqsubseteq} A_2$ by abstracting both the domain and the codomain: $\alpha_\varphi(F^\sharp) = \alpha_2 \circ F^\sharp \circ \gamma_1$, $\gamma_\varphi(F^{\sharp\sharp}) = \gamma_2 \circ F^{\sharp\sharp} \circ \alpha_1$. One can also abstract separately the domain (abstraction δ) and the codomain (pointwise abstraction π) to obtain two intermediate abstractions.



(a) for sets of functions (b) for relations

$L_1 \simeq L_2$ means lattice isomorphism
$L_1 \longrightarrow L_2$ means that $L_1 \succeq L_2$ (L_2 abstracts L_1)
$L_1 \Longleftarrow L_2$ means that L_1 is the disjunctive completion of L_2.

Fig. 1. Lattice of abstract domains for functions and relations.

Fig. 2 shows also that viewing a function $D_1 \to D_2$ as a Boolean function $D_1 \times D_2 \to \mathbb{B}$, as canonical abstraction [5] does, leads $\simeq$ better precision.

$$
\begin{array}{lccccc}
\text{Viewing a function as:} & D_1 \times D_2 \to \mathbb{B} & & D_1 \to D_2 & & \wp(D_1 \times D_2) \\
\text{Abstracting sets of functions with:} & \downarrow \phi & & \downarrow \phi & & \downarrow [\rho, \vee] \\
\text{Resulting abstract domains:} & A_1 \times A_2 \to \wp(\mathbb{B}) & \succeq & A_1 \to A_2 & \succeq & A_1 \times A_2
\end{array}
$$

Fig. 2. Different ways of coding (sets of) functions, and the resulting abstractions.

Relational function-abstraction. The principle of relational function-abstraction is illustrated in Fig. 3. In this figure, we abstract by different methods the set of functions $A1$, and we observe in $A2$, $A3$ and $A4$, which are images of $A1$ by different $\gamma \circ \alpha$ mappings, the loss of information induced by these methods.

The point of our abstraction is to avoid the abstraction ϱ, which loses (in D) the relation $f(u_1) = f(u_2)$ that holds for any $f \in A1$. Instead, we directly abstract the domain U separately on each element of $A1$, obtaining $B1$. This is equivalent to the disjunctive completion of $\alpha_\delta \circ \alpha_\varrho$.

We then apply the original abstraction η to obtain C. The point is that if $U^\sharp$ is finite, C is a set of $|U^\sharp|$-dimensional vectors and can in turn be abstracted by any numerical lattices (octagons, polyhedra). The definition of η, and particularly

γ_η, is somewhat subtle:

$$\alpha_\eta(F) = \bigcup_{f \in F} \{f^\sharp \mid \forall u^\sharp : f^\sharp(u^\sharp) \in f(u^\sharp)\} \tag{1}$$

$$\gamma_\eta(F^\sharp) = \{f \mid \forall f^\sharp \in (U^\sharp \to \mathbb{R}) : \left(\forall u^\sharp, f^\sharp(u^\sharp) \in f(u^\sharp)\right) \Rightarrow f^\sharp \in F^\sharp\} \tag{2}$$

Another formulation can be given by treating functions from (finite) $U^\sharp$ as vectors:

$$\alpha_\eta(F) = \bigcup_{\langle X_1,\ldots,X_n\rangle \in F} X_1 \times \ldots \times X_n \tag{3}$$

$$\gamma_\eta(F^\sharp) = \{\langle X_1, \ldots, X_n\rangle \mid X_1 \times \ldots \times X_n \subseteq F^\sharp\} \tag{4}$$

A comparison of $B2$ with $B1$ shows what is lost by η. Observe, however, that the relation $f(u_1^\sharp) = f(u_2^\sharp)$ is preserved. A comparison of $A3$ with $A4$ shows what is gained with relational function-abstraction Φ compared to the classical abstraction ϕ. For instance, we preserve in $A3$ the property $f(u_2) + 2 \leq f(u_3) \leq f(u_2) + 3$, unlike in $A4$.

Because γ_η is not trivial, our contribution can also be viewed as a way to give a meaning to a set $|U^\sharp|$-dimensional vectors as a set of functions $U \to \mathbb{R}$, where U may be infinite, using the concretization $\gamma^\vee_{\varrho,\delta} \circ \gamma_\eta$.

In full generality, if the codomain is also abstracted, assuming $\alpha_2[n]$ abstracts $\wp(D_2^n)$ with $A_2[n]$, relational function-abstraction α_Φ is defined as $\alpha_2[n] \circ \alpha_\eta \circ \alpha^\vee_{\varrho,\delta}$.

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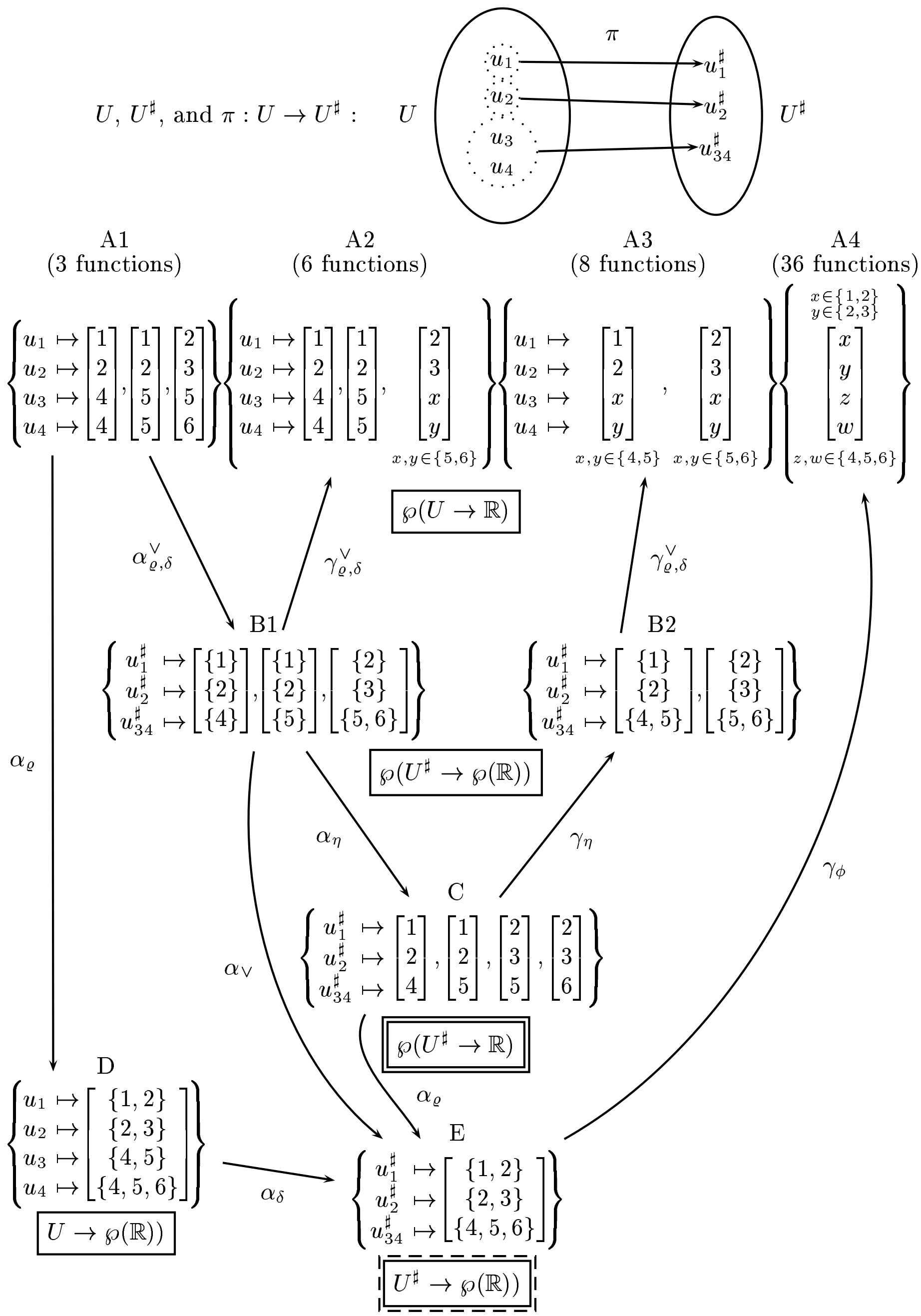

Fig. 3. Different abstractions of the concrete set of functions A1 and the loss of information induced by them (shown by concrete values A2, A3, and A4). Abstract value C (whose concretization is A3) is the abstract value obtained with relational function-abstraction Φ, whereas abstract value E (whose concretization is A4) is the abstract value obtained by the abstraction ϕ.